

Equivalent electrical network model approach applied to a double acting low temperature differential Stirling engine



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ABSTRACT

This work presents a network model to simulate the periodic behavior of a double acting free piston type Stirling engine. Each component of the engine is considered independently and its equivalent electrical circuit derived. When assembled in a global electrical network, a global model of the engine is established. Its steady behavior can be obtained by the analysis of the transfer function for one phase from the piston to the expansion chamber. It is then possible to simulate the dynamic (steady state stroke and operation frequency) as well as the thermodynamic performances (output power and efficiency) for given mean pressure, heat source and heat sink temperatures. The motion amplitude especially can be determined by the spring-mass properties of the moving parts and the main nonlinear effects which are taken into account in the model. The thermodynamic features of the model have then been validated using the classical isothermal Schmidt analysis for a given stroke. A three-phase low temperature differential double acting free membrane architecture has been built and tested. The experimental results are compared with the model and a satisfactory agreement is obtained. The stroke and operating frequency are predicted with less than 2% error whereas the output power discrepancy is of about 30%. Finally, some optimization routes are suggested to improve the design and maximize the performances aiming at waste heat recovery applications.

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1. Introduction

Low temperature differential (LTD) Stirling engines are machines that can operate with hot source temperature of about 150 °C. At this operating temperature and assuming a sink temperature of 25 °C, the Carnot efficiency is 29%. Then, a Stirling generator which would achieve 50% of this efficiency might be of interest for power generation. Solar powered LTD Stirling engine appears to be a promising technology [1] and is also a potential technology for waste heat recovery [2,3]. In both applications, the simplicity and reliability of the Stirling machines are significant advantages toward the development of low cost generators.

Among the various potential Stirling architectures, the double acting engines type also called Rinia or multiphase architecture [4,5] is of particular interest for the aforementioned applications. In such configuration, three, four or more alpha type engines (also called phases) are connected to each other. The piston of the engine $i-1$ acts as the displacer of its neighboring engine i . In [6], Abdullah studied such a double acting LTD Stirling. He underlines that the specific material and components required for the

crankshaft, the connecting rods and the gaskets are an important issue for LTD applications.

A free piston Stirling engine (FPSE) architecture allows avoiding complex mechanical linkages and the resulting robustness and reliability questions. Though, their optimization has been proven to be difficult especially for the piston-displacer phase angle control [7,8]. The double acting architecture overcomes this difficulty because the phase angle is fixed by the number of phases. It equals 120° in the case of three engines and 90° in the case of four engines. Therefore, the free piston double acting arrangement appears to be a great advantage compared to the usual FPSE. However, a proper sealing of the piston and the displacer can be difficult to ensure.

This last obstacle can be classically solved using membranes. In his work, Minassians proposed a LTD double acting FPSE using membranes instead of pistons [9,10]. The operation has been demonstrated on a $3 \times 350 \text{ cm}^3$ total volume engine. A theoretical efficiency of 19.7% which is about 70% of the Carnot efficiency was expected to be reached using helium or pressurized air engine.

In the case of double acting FPSE, the strong coupling between dynamic and thermodynamic has to be addressed properly. The major non-linear effects have to be integrated in the model to predict the performances. These effects are usually associated to the gas friction losses within the components and the mechanical non-linearity [7,8,11–13].

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Nomenclature

A_d	displacer area (m ²)
A_p	piston area (m ²)
A_{ff}	free flow area (m ²)
A_w	wetted area (m ²)
c_d	displacer mechanical dissipation (Nm ⁻¹ s)
c_p	piston mechanical dissipation (Nm ⁻¹ s)
C_f	friction factor (–)
D	damping coefficient (Nm ⁻¹ s)
d_w	wire diameter of the matrix regenerator (m)
d_{hi}	hydraulic diameter of component i (m)
D_d	displacer diameter (m)
D_p	piston diameter (m)
E_b	young modulus (Nm ⁻²)
f	frequency (Hz)
f_f	friction factor (–)
f_m	mechanical force (N)
I_b	second moment of the spring beam (m ⁴)
j	complex number such as $j^2 = -1$
k	gas thermal conductivity (Wm ⁻¹ K)
K	spring beam stiffness (Nm ⁻¹)
L_i	length of exchanger (m)
L_b	length of the spring beam (m)
m_{tot}	total mass of gas (kg)
M	mass of a rigid link (kg)
m_i	mass of gas within component i (kg)
n_i	number of channels of component i (–)
P_m	mechanical power (W)
p	pressure (N m ⁻²)
r	ideal mass gas constant (J kg ⁻¹ K ⁻¹)
S_b	cross-section of the spring beam (m ²)
T_{cold}	cold temperature (K)

T_{hot}	hot temperature (K)
T_i	temperature of gas within component i (K)
T_w	wall temperature (K)
u_i	mean gas velocity in component i (ms ⁻¹)
V_i	volume of component i (m ³)
V_{sw}	swept volume (m ³)
V_{swe}	expansion space swept volume (m ³)
V_{swc}	compression space swept volume (m ³)

Greek symbols

α	swept volume phase angle (rad)
χ	fluid flow resistance coefficient (s ⁻¹)
Δp	pressure loss (N m ⁻²)
ε	scale (–)
κ	swept volume ratio (–)
μ	dynamic viscosity (N m ⁻² s)
ρ	gas density (kg m ⁻³)
ρ_b	material density (kg m ⁻³)
ω	pulsation (rad s ⁻¹)
φ	porosity of the regenerator (–)

Subscripts

b	beam
c	compression space
d	displacer
e	expansion space
h	heater
k	cooler
p	piston
r	regenerator

In [9] Minassians proposed an isothermal dynamic model to study the behavior of the multiphase engine. Based on the Schmidt analysis, the instantaneous pressure is expressed as a function of the piston positions. Then the mechanical equilibrium equation of each piston is established including the dissipation forces and the external load. A linearization strategy allows an analytical expression of the start-up condition to be obtained. Above this critical value, a numerical resolution has to be used to obtain the periodic motion of the engine and to evaluate its performance. The flow friction and hysteresis losses have been pointed out because of their effects on the engine performances and have to be taken into account for design of such LTD engines.

Another modeling approach based on the control system theory of a wobble-yoke double acting Stirling engine has been presented in [14,15]. The engine architecture implies some flexibility in the mechanical connections of the pistons so its operation is close to double acting FPSE. The modeling strategy relies on the dynamical equilibrium equations established for each of the piston. The pressure can then be defined as a function of the pistons' position using the Schmidt analysis [4] (perfect gas law, isothermal transformations). A linearization is performed to obtain a state space representation of the system. Such an approach leads to an unstable dynamics so a control concept based modification namely a pre-compensator, is defined to reproduce the pistons' amplitudes. This model has not been compared to experimental results and some important losses such as flow friction are not taken into account.

In this paper, a generic network model is proposed and validated to study the dynamic and thermodynamics behavior of a double acting free membrane Stirling engine.

As a basis for the study, a novel configuration of a LTD double acting FPSE derived from the work of Minassians [10] is

introduced. Fig. 1a presents the global engine. It is composed of three alpha type engines using membranes and beam springs which are connected to each other via rigid links. The top side of the system can be connected to the heat source whereas the bottom side is cooled. For each engine, the simplest architecture has been sought here. So, the heater and cooler, the expansion and compression chambers are identical (see Fig. 1b). The heater and cooler consist in 20 holed punched plates fitted together in a total thickness $L_h = L_k$. By doing this, the axial conduction through the solid material is impeded whereas the radial conduction allows aiming at isothermal processes in the heat exchanger volumes. The regenerator consists in a stack of 40 plain weave wire meshes. The wire diameter is 50 μ m and apertures are 350 μ m squares. The main dimensions and characteristics of the engine are given in Table 1.

In a first part, the equivalent electrical model for each component of the engine is detailed. The classical electrical analogy is chosen: the pressure is seen as the electrical voltage and the volume flow rate as the electrical current. The continuity and momentum equations are established for the compression and expansion chambers, the heat exchangers and the regenerator. Then using linear approximation for the time and space variables, the equivalent electrical circuits are derived. Finally, the electrical circuits are connected to each other to obtain the dynamic equations. A global analysis is then achieved to establish the start-up condition and evaluate the performances. Then, as a first validation step, the thermodynamic results of the proposed model are compared with those obtained using the Schmidt analysis. The effect of the load on the critical start-up temperature and the engine performances is studied. An experimental prototype has been built and is presented in the last part. The experimental measurements are

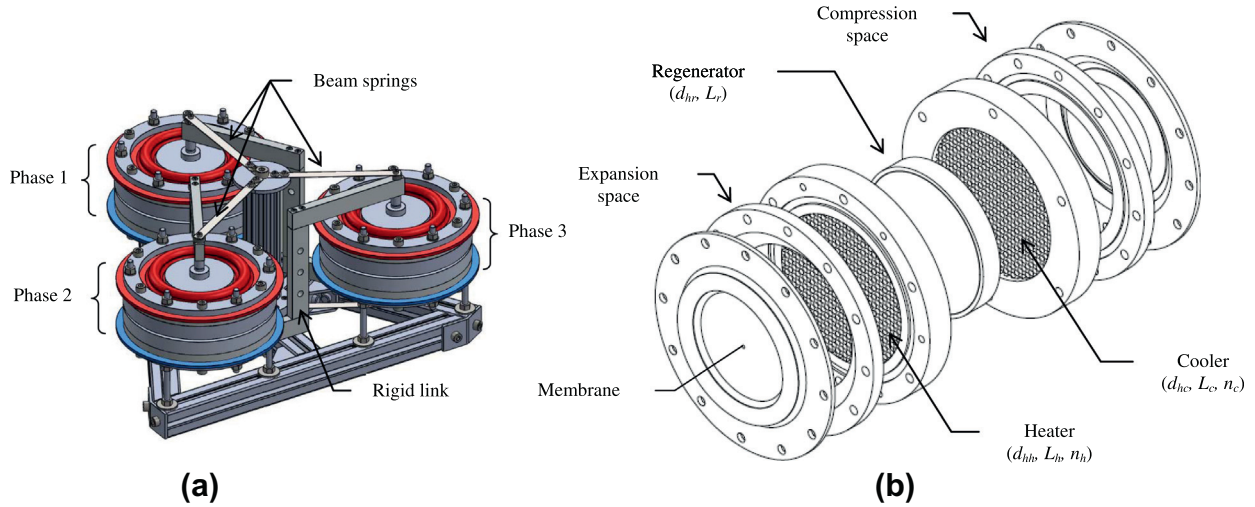


Fig. 1. (a) Schematic of the proposed double acting free-piston engine and (b) exploded view of a single phase.

Table 1

Main parameters values of the studied double acting Stirling engine.

Volumes for one engine						
	V_c	164.4 cm ³	V_e	164.4 cm ³	V_{tot}	440 cm ³
Heat exchangers and regenerator						
Hydraulic Ø	d_h	4 mm	d_k	4 mm	d_r	0.28 mm
Length	L_h	15.5 mm	L_k	15.5 mm	L_r	15.2 mm
Beam springs						
Material = Steel	L_b	100 mm	S_b	5.6 mm ²	I_b	0.146 mm ⁴
Membrane						
Material = silicone	A_p	4.3 cm ²	A_d	4.3 cm ²		
Typical temperatures and mean pressure						
	T_c	25–40 °C	T_h	140–160 °C	$\bar{p}$	Atmospheric pressure

compared with the proposed model and the discrepancies are studied to improve the prototype and appraise the modeling strategy. Based on the presented prototype, technological developments are suggested to improve the design and enhance the performances aiming at waste heat recovery applications.

2. Network modeling of each component

In the following the components of one engine (or phase) are addressed successively. The governing equation are set and linearized to obtain an equivalent electrical model.

2.1. Compression and expansion spaces

The compression and expansion spaces are the volumes swept by the membranes. An isothermal assumption is chosen here.

The continuity equation for a perfect gas within a chamber leads to:

$$\dot{m}_j(t) = \frac{p_j(t)}{rT_j(t)} \dot{V}_j(t) + \frac{V_j(t)}{rT_j(t)} \dot{p}_j(t) \quad (1)$$

In which $j = c, e$ for the compression and expansion spaces respectively.

The variations of the pressure, the mass and the volume are assumed to be small with respect to their mean value over a cycle and are denoted $\bar{p}$, $\bar{m}_j$ and $\bar{V}_j$ respectively. Then, we can set: $p_j(t) = \bar{p} + \tilde{p}_j(t)$, $m_j(t) = \bar{m}_j + \tilde{m}_j(t)$, $V_j(t) = \bar{V}_j + \tilde{V}_j(t)$. Neglecting high order terms, Eq. (1) is written again as:

$$\dot{\tilde{m}}_j(t) = \frac{\bar{p}}{rT_j} \dot{\tilde{V}}_j(t) + \frac{\bar{V}_j}{rT_j} \dot{\tilde{p}}_j(t) \quad (2)$$

The mass flow rate out of the considered volume is: $\dot{\tilde{m}}_j(t) = -\bar{\rho}_j u_j(t)$ in which the local density is $\bar{\rho}_j = \frac{\bar{p}}{rT_j}$ and $u_j(t)$ is the volumetric flow rate out of the chamber. $u_p(t) = -\dot{\tilde{V}}_j(t)$ is the instantaneous volume swept by the membrane and is then positive whenever the volume decreases ($\dot{\tilde{V}}_j(t) < 0$). Therefore, Eq. (2) is finally expressed as:

$$u_j(t) = u_p(t) - C_j \dot{\tilde{p}}_j(t) \quad (3)$$

In which $C_j = \frac{\bar{V}_j}{\bar{p}}$ is an equivalent electrical capacitance. Eq. (3) can then be read as the electrical circuit of Fig. 2a.

2.2. Heat exchanger

We assume that the inertia of the gas is neglected and the temperature is constant throughout the heat exchanger. The local

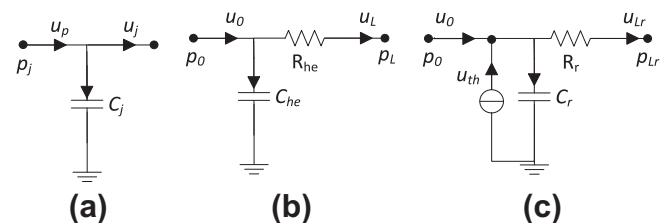


Fig. 2. Equivalent electrical circuits for: (a) the expansion or compression chamber, (b) the heat exchanger and (c) the regenerator.

equations for the conservation of mass and momentum for 1D flow are then:

$$\frac{1}{rT_q} \frac{\partial p_q(x, t)}{\partial t} + \frac{1}{A_{ffq}} \frac{\partial \dot{m}_q(x, t)}{\partial x} = 0 \quad (4)$$

$$\frac{\partial p_q(x, t)}{\partial x} + \frac{\chi}{A_{ffq}} \dot{m}_q(x, t) = 0 \quad (5)$$

In which $q = k, h$ for the cooler and the heater respectively.

Applying the same approximation as before and retaining the lowest order terms, we obtain:

$$\frac{1}{rT_q} \frac{\partial \tilde{p}_q(x, t)}{\partial t} + \frac{1}{A_{ffq}} \frac{\partial \dot{\tilde{m}}_q(x, t)}{\partial x} = 0 \quad (6)$$

$$\frac{\partial \tilde{p}_q(x, t)}{\partial x} + \frac{\chi}{A_{ffq}} \dot{\tilde{m}}_q(x, t) = 0 \quad (7)$$

In the Laplace space, the previous Eqs. (6) and (7) can be integrated for the geometric variable x to bring:

$$\begin{bmatrix} \dot{\tilde{m}}_q(x, s) \\ \tilde{p}_q(x, s) \end{bmatrix} = \begin{bmatrix} \cosh(\Gamma(s)x) & -\frac{1}{Z_c(s)} \sinh(\Gamma(s)x) \\ -Z_c(s) \sinh(\Gamma(s)x) & \cosh(\Gamma(s)x) \end{bmatrix} \begin{bmatrix} \dot{\tilde{m}}_q(0, s) \\ \tilde{p}_q(0, s) \end{bmatrix} \quad (8)$$

In which $\Gamma(s) = \sqrt{s \frac{\chi}{rT_q}}$, $Z_c(s) = \frac{1}{A_{ffq}} \sqrt{\frac{1}{s} \chi r T_q}$ and s is the Laplace transformation variable.

Retaining the lowest order terms of the power series expansions of the cosh and the sinh functions, a spatially independent model is derived from Eq. (8). The governing equations for the heat exchangers are then:

$$\dot{\tilde{m}}_q(L_q, s) = \dot{\tilde{m}}_q(0, s) - \frac{A_{ffq} L_q}{r T_q} s \tilde{p}_q(0, s) \quad (9)$$

$$\tilde{p}_q(L_q, s) = \tilde{p}_q(0, s) - \frac{\chi L_q}{A_{ffq}} \dot{\tilde{m}}_q(0, s) \quad (10)$$

Replacing Eq. (9) in Eq. (10), the relationship between the input and output pressures can then be expressed as:

$$\tilde{p}_q(L_q, s) = \left(1 - s \frac{\chi L_q^2}{r T_q}\right) \tilde{p}_q(0, s) - \frac{\chi L_q}{A_{ffq}} \dot{\tilde{m}}_q(L_q, s) \quad (11)$$

Retaining the first order terms, Eq. (11) is finally simplified to:

$$\tilde{p}_q(L_q, s) = \tilde{p}_q(0, s) - \frac{\chi L_q}{A_{ffq}} \dot{\tilde{m}}_q(L_q, s) \quad (12)$$

Using the volumetric flow rate and the time variable, Eqs. (9) and (12) are rewritten as:

$$u_{Lq}(t) = u_{0q}(t) - C_q \dot{\tilde{p}}_{0q}(t) \quad (13)$$

$$\tilde{p}_{Lq}(t) = \tilde{p}_{0q}(t) - \frac{\chi L_q}{A_{ffq}} \bar{\rho}_q u_{Lq}(t) \quad (14)$$

In which $C_q = \frac{A_{ffq} L_q}{p}$ is the equivalent capacitance and $\bar{\rho}_q = \frac{\bar{p}_q}{r T_q}$.

2.2.1. Modeling of the pressure drop

The fluid flow resistance coefficient χ can be derived from the pressure drop – Reynolds number relationship and is related to the friction factor f_f by Eq. (15):

$$\chi = 8 \frac{f_f}{\pi d_{hq}^3} |u_{Lq}(t)| \quad (15)$$

In which f_f is in the form $f_f = C_f \text{Rey}^n$ in which $\text{Rey} = 4 \frac{\bar{\rho}_q}{\pi d_{hq} \mu} u_{Lq}(t)$ is the instantaneous Reynolds number.

Using Eq. (15) in Eq. (14), we obtain the expression of the pressure drop across the heat exchanger:

$$\tilde{p}_{Lq}(t) = \tilde{p}_{0q}(t) - \frac{2^{2n+5}}{\pi^{n+2}} C_f \frac{\bar{\rho}_q^{n+1}}{\mu^n} \frac{L_q}{d_{hq}^{5+n}} |u_{Lq}(t)|^{n+1} u_{Lq}(t) \quad (16)$$

Eq. (16) is nonlinear and it is useful to anticipate the value of the friction factor C_f for the operating conditions of the engine for further simplification.

LTD Stirling engines usually presents low operating frequency and small swept volume. Assuming millimeter range strokes as well as operating frequency below 100 Hz, the mean Reynolds number can be approximated using the model proposed by Organ [16] and used in [7]. Fig. 3 presents the estimation of the Reynolds numbers in the heat exchangers and the regenerator for membrane strokes of 3 mm, $T_e = 130^\circ\text{C}$, $T_c = 50^\circ\text{C}$ and a phase angle of 120° .

Following Martini's approximation for circular tubes [17], Reynolds numbers less than 2000 lead to $C_f = 16$ and $n = -1$.

In these conditions, Eq. (16) can be simplified to:

$$\tilde{p}_{Lq}(t) = \tilde{p}_{0q}(t) - R_q u_{Lq}(t) \quad (17)$$

In which $R_q = 128 \frac{\mu L_q}{\pi d_{hq}^3}$ is the equivalent constant resistance.

Finally, Eqs. (13) and (17) can then be seen as the electrical circuit of Fig. 2b. The evaluation of the Reynolds number from the model results will be performed and this assumption validated.

2.3. Regenerator

Following the same strategy as for the heat exchangers, the governing equations for the regenerator are:

$$\dot{\tilde{m}}_r(L_r, s) = \dot{\tilde{m}}_r(0, s) - \frac{A_{ffr} L_r}{r T_{rmean}} s \tilde{p}_r(0, s) \quad (18)$$

$$\tilde{p}_r(L_r, s) = \tilde{p}_r(0, s) - \frac{\chi L_r}{A_{ffr}} \dot{\tilde{m}}_r(L_r, s) \quad (19)$$

In which $T_{rmean} = \frac{T_h - T_c}{\log(T_h/T_c)}$.

An electrical equivalent circuit is finally completed using the volumetric flow rate:

$$u_{Lr}(s) = \frac{T_h}{T_c} u_{0r} - C_r s \tilde{p}_r(0, s) \quad (20)$$

$$\tilde{p}_{Lr}(s) = \tilde{p}_0(s) - \frac{\chi L_r}{A_{ffr}} \bar{\rho}_r u_{Lr}(s) \quad (21)$$

In which $C_r = \frac{T_h}{T_{rmean}} \frac{A_{ffr} L_r}{p}$ is the equivalent capacitance. Eq. (20) can be written again considering an additional “current source” to take into account the temperature gradient in the regenerator:

$$u_{Lr}(s) = u_{0r}(s) - C_r s \tilde{p}_r(0, s) + \left(\frac{T_h}{T_c} - 1\right) u_{0r}(s) \quad (22)$$

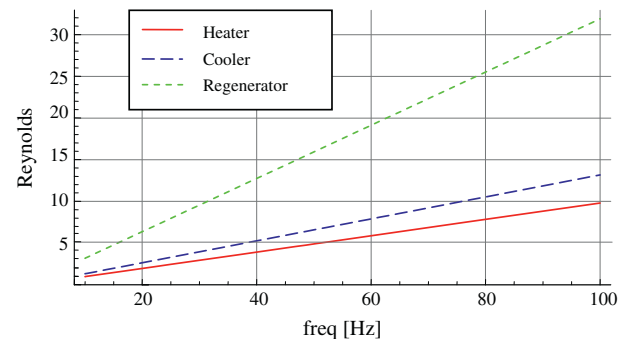


Fig. 3. Estimation of the Reynolds numbers for the heat exchangers and the regenerator as a function of the operating frequency.

The last term of Eq. (22) is a thermal source flow which will be denoted $u_{th}(s)$ in the following defining $u_{th}(s) = Gu_{or}(s)$ and $G = (\frac{T_h}{T_c} - 1)$. Fig. 2c presents the equivalent electrical circuit.

2.3.1. Modeling of the pressure drop

As for the heat exchangers, the resistance coefficient χ is related to the friction factor and the Reynolds number. For a woven screen matrix regenerator the hydraulic diameter is a function of the wire diameter and the porosity such as: $d_h = d_w \frac{1}{1-\epsilon}$. For the regenerator, the estimated Reynolds number is smaller than 30 as can be seen in Fig. 3. Then, the proposed experimental correlation is obtained from [18]: $C_f = 3.63$ and $n = -0.84$. The dissipation term in Eq. (21) is finally a nonlinear expression with respect to the volumetric flow rate $u_{Lr}(t)$.

A quadratic polynomial approximation is used so the equivalent resistance R_r is defined as:

$$R_r = \frac{1}{2} C_f \frac{\bar{\rho}_r^{n+1}}{\mu^n} \frac{L_r}{d_{hr}^{1-n} A_{jfr}^{2+n}} (a(n) + b(n) u_{Lr}(t)^2) \quad (23)$$

In which $a(n)$ and $b(n)$ are functions of the coefficient n .

A numerical evaluation in the case of the proposed Stirling show that the quadratic term $b(n)$ is several orders of magnitude lower than the constant term $a(n)$. Therefore, only the latter will be retained in the simulation leading to a linear damping effect. Eqs. (21) and (22) can then be presented as the equivalent electrical circuit of Fig. 2c.

2.4. Beam spring stiffness and rigid arm dynamics

The dynamics of the engine partly relies on the beam spring and the mass of the rigid links which connect the membranes together (see Fig. 1a), whereas the silicone membranes mostly induce mechanical dissipations.

2.4.1. Beam spring stiffness

Beam springs are clamped to the rigid link at one end and to the frame of the engine at the other side. The stroke of the engine is bounded by the strongest nonlinearity of the engine [13]. Then, large displacements of the beam are to be considered and may be the highest nonlinear effect because the pressure drop effects are expected to be small (low Reynolds numbers).

Following the modeling strategy of Azrar [19], a quasi-static model based on the Lagrange's equations is performed to obtain the large deformation stiffness of each spring beam. Using a single mechanical eigenmode approximation, the displacement of the tip of the beam can be written as:

$$w(L_b, t) = q_1(t) \phi_1(L_b) \quad (24)$$

The static equilibrium equation of one beam spring is given by Eq. (25):

$$k_{11} q_1(t) + 2b_{1111} q_1(t)^3 = F_1 f_m(t) \quad (25)$$

In which, $k_{11} = E_b I_b \int_0^{L_b} \frac{d^2 \phi_1(x)}{dx^2} \frac{d^2 \phi_1(x)}{dx^2} dx$, $b_{1111} = \frac{E_b S_b}{4 I_b} \int_0^{L_b} \frac{d \phi_1(x)}{dx} \frac{d \phi_1(x)}{dx} dx$, $F_1 = \phi_1(L_b)$ and $f_m(t)$ any additional force applied to the tip of the beam.

From Eqs. (24) and (25), the nonlinear stiffness of the beam spring is defined as:

$$K(w(L_b, t)) = k_{11} + \frac{2b_{1111}}{\phi_1(L_b)^2} w(L_b, t)^2 \quad (26)$$

In order to validate the proposed semi-analytical approach, a FEM model of the beam has been performed and the stiffness evaluated. Fig. 4 shows the evolution of the nonlinear stiffness as a function of the tip of the beam displacement for the FEM and the

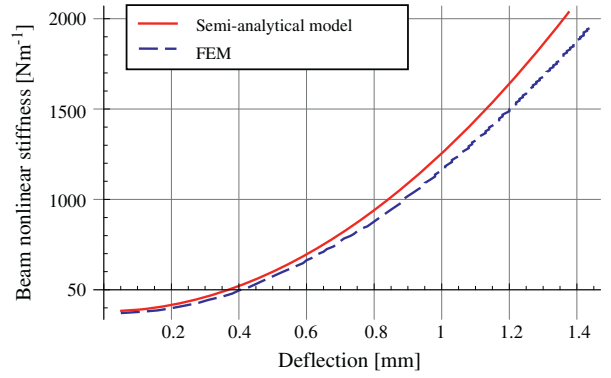


Fig. 4. Evolution of the nonlinear spring beam stiffness as a function of the deflection for the beam dimensions given in Table 1.

semi-analytical model. Though a moderate discrepancy can be seen between the two models, the semi-analytical model is validated and precise enough for the proposed engine modeling strategy.

2.4.2. Rigid link dynamics

The dynamic equilibrium of each rigid arm linked to the piston and displacer membranes of two successive phases can be written as in Eq. (27):

$$M \frac{d^2 w(L_b, t)}{dt^2} + D \frac{dw(L_b, t)}{dt} + 2K(w(L_b, t))w(L_b, t) = A_p p_{i-1}(t) - A_d p_i(t) \quad (27)$$

D is the damping coefficient which represents the mechanical losses within the membranes and any additional load ($D = D_{mec} + D_{load}$).

Recalling Eq. (3), the volumetric flow rate related to the displacement of the tip of the beam is $u_p(t) = -A_p \dot{w}(L_b, t)$ for the piston membrane ($u_d(t) = A_d \dot{w}(L_b, t)$ for the displacer membrane). Moreover, because the piston and displacer areas are the same, Eq. (27) can finally be written as:

$$L_p \frac{du_p(t)}{dt} + R_p u_p(t) + \frac{1}{C_p(u_p)} \int u_p(t) = p_i(t) - p_{i-1}(t) \quad (28)$$

In which $L_p = \frac{M}{A_p^2}$, $R_p = \frac{D}{A_p^2}$ and $C_p(u_p) = \frac{A_p^2}{2K(u_p)}$ are the equivalent inductance, resistance and nonlinear capacitance respectively.

The electrical circuit of Fig. 5 is then consistent with Eq. (28).

The nonlinearity of the stiffness term in Eq. (27) (i.e. the capacitive term in Eq. (28)) as to be taken into account properly for the system analysis in the Laplace space. Numerous methods can be used to obtain analytical solutions for the dynamic of nonlinear oscillators as described in [20]. The approximate first harmonic solution of free oscillations of Eq. (28) when $p_{i-1}(t) = p_i(t) = 0$ can be written as:

$$u_p(t) = u_{p0} \cos(\omega_p t) = u_{p0} \cos \left(\sqrt{\alpha_1} \left(1 + \frac{3}{8} \frac{\alpha_3}{\alpha_1} u_{p0}^2 \right) t \right) \quad (29)$$

In which $\alpha_1 = 2 \frac{k_{11}}{M}$, $\alpha_3 = 4 \frac{b_{1111}}{\phi_1(L_b)^2} \frac{1}{MA_p^2}$ and u_{p0} is the initial amplitude condition of the differential equation.

Eq. (29) exhibits the characteristic frequency amplitude dependent behavior of nonlinear cubic oscillator. The natural oscillation frequency is a function of the amplitude of the response. Therefore,

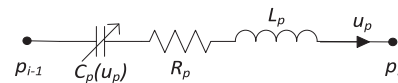
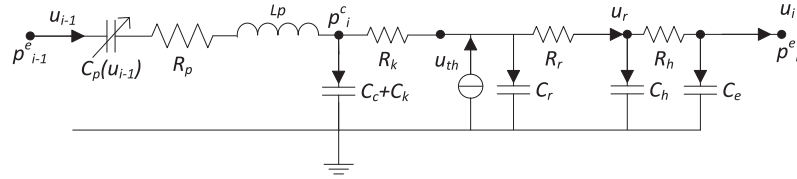


Fig. 5. Electrical circuit equivalent to the dynamics of the membrane and beam spring.

Fig. 6. Global network model of the engine i .

we approximate the Laplace transform as written in Eq. (30) in which the “natural” frequency is dependent on the amplitude of the volumetric flow rate U_p :

$$p_{i-1}^e = \frac{[1 + sZ_p(C_c + C_r) + G(1 + sC_H R_r)(1 + sC_C Z_p) + sC_H(R_r + Z_p + sR_r Z_p(C_c + C_r))]}{s[C_H + C_r + sC_H C_r R_r + C_c(1 + G)(1 + sC_H R_r)]} u_{i-1} - u_i \quad (33)$$

$$\left(L_p s^2 + R_p s + L_p \alpha_1 \left(1 + \frac{3}{8} \frac{\alpha_3}{\alpha_1} U_p^2 \right) \right) U_p = s(p_i(s) - p_{i-1}(s)) \quad (30)$$

2.5. Hysteresis loss

The compression and expansion spaces act as gas springs and hysteresis loss is considered within these spaces as it was pointed out in Minassian's study [9]. Following the approximation of Urieli [4], the average value of the gas spring hysteresis loss power for the compression or the expansion space is:

$$\dot{W}_{hyst} = \sqrt{\frac{1}{32} \omega \gamma^3 (\gamma - 1) T_w \bar{p} k} \left(\frac{\Delta \tilde{V}_j}{\tilde{V}_j} \right)^2 A_w \quad (31)$$

In which $\Delta \tilde{V}_j$ is the average swept volume.

Therefore, a linear damping can be inferred from Eq. (31) and is defined as [4]:

$$D_{hyst} = \sqrt{\frac{1}{8} \frac{1}{\omega^3} \gamma^3 (\gamma - 1) T_w \bar{p} k} \frac{1}{\tilde{V}_j^2} A_w \quad (32)$$

This damping effect can be added to the mechanical dissipation. Consequently, the equivalent resistance R_p of Eq. (28) is modified such as the hysteresis loss is taken into account and the equivalent electrical resistance is: $R_p = \frac{D}{A_p^2} + D_{hyst}$.

3. Global model: coupled thermodynamic-dynamic analysis

The modeling of each of the elements of one free piston engine i has been performed. Using the continuity of the pressure and the volumetric flow rate from one element to the other, a global electrical network is derived and is shown in Fig. 6. The pressure p_{i-1}^e at the left hand side of the scheme stands for the expansion chamber pressure of the engine $i - 1$ whereas p_i^e is the expansion chamber pressure of the considered engine i .

From this equivalent network, the relationships between the input pressure p_{i-1}^e and the volumetric flow rate u_{i-1} and the output p_i^e and u_i can be established.

In the following, the pressure drops through the heat exchangers is neglected compared to the one across the regenerator. Therefore, the equivalent electrical resistances R_k and R_h are supposed to be zero and the scheme of Fig. 6 is simplified accordingly and is presented in Fig. 7.

In the Laplace space, the input–output relationships are then expressed in Eqs. (33) and (34):

$$p_i^e = \frac{(1 + G)u_{i-1} - (1 + s(C_r R_r + C_c(1 + G)R_r))u_i}{s[C_H + C_r + sC_H C_r R_r + C_c(1 + G)(1 + sC_H R_r)]} \quad (34)$$

In which $Z_p = \frac{1}{sC_p} + R_p + sL_p$, $C_c = C_c + C_k$ and $C_H = C_e + C_h$.

3.1. Solution procedure

For each of the engine, two equations similar to Eqs. (33) and (34) are obtained. Considering all the phases, a total of 6 equations for 6 unknown variables can be written. The resolution of this non-linear parametric system allows the magnitude and frequency of operation to be found.

If we assume three identical phases, the condition of operation can be simply expressed as:

- A three-phase double acting Stirling operation presents a 120° phase angle between the motions of each piston and displacer.
- The magnitude of the displacement of the piston of the engine i equals the magnitude of the piston of the engine $i - 1$.

Therefore, the following relationship is set between the input and output volumetric fluid flows u_{i-1} and u_i respectively:

$$u_i = \exp(-j\varphi)u_{i-1} \quad (35)$$

In which $\varphi = 120^\circ$ for of a three-phase engine operation.

Replacing Eq. (35), into Eqs. (33) and (34) allows the expression of the output pressure p_i^e to be expressed as a function of the input pressure p_{i-1}^e such as $p_i^e = H_{pi}(j\omega)p_{i-1}^e$. The operation conditions are then:

$$H_{pi}(j\omega)\angle = -\frac{2\pi}{3} \quad (36)$$

$$|H_{pi}(j\omega)| = 1 \quad (37)$$

The phase condition Eq. (36) determines the oscillation frequency ω whereas the stroke is defined by the magnitude condition Eq. (37).

For $T_h = 120^\circ \text{C}$, $T_c = 40^\circ \text{C}$, the magnitude of the transfer function $H_{pi}(j\omega)$ at the 120° phase angle, is 1.027 at 35.36 Hz for 0 mm stroke (case A, dotted line in Fig. 8). A limit cycle is not reached as condition Eq. (37) is not fulfilled. Still, the magnitude

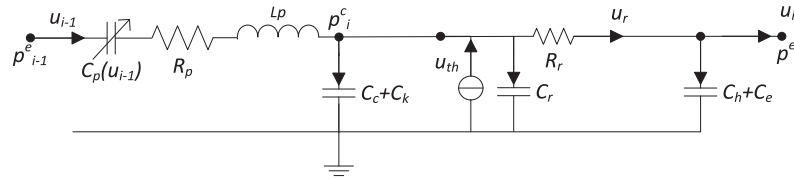
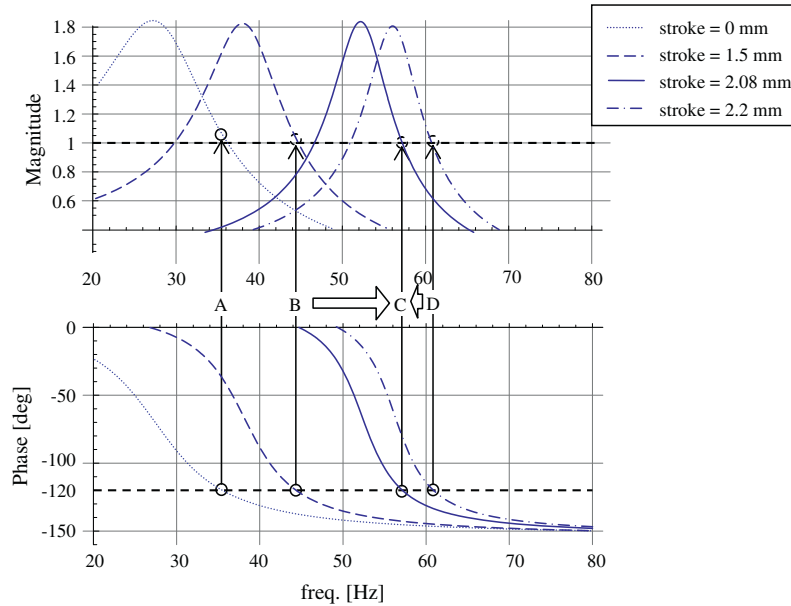


Fig. 7. Simplified global network model of the engine i.

Fig. 8. Magnitude and phase plots for $H_{pi}(j\omega)$ for different strokes.

of $H_{pi}(j\omega)$ is greater than unity which indicates that the amplitude will grow with time.

Higher strokes imply an increased stiffness of the beam at maximal amplitude and induce higher operating frequencies eventually. As an example, for a 1.5 mm stroke the frequency for which $H_{pi}(j\omega) \angle = \frac{2\pi}{3}$ is 44.33 Hz (case B, dashed line in Fig. 8).

The limit cycle occurs at 57.35 Hz when the conditions of Eqs. (36) and (37) are reached simultaneously (case C, plain line in Fig. 8). In this case, the stroke is 2.08 mm.

For higher stroke, the magnitude of the transfer function decreases. For 2.2 mm stroke, the magnitude is 0.994 at 60.91 Hz (case D, dashed-dotted line in Fig. 8). This proves that a limit cycle is reached for the case C.

3.2. Results and discussion

3.2.1. Energy and power balance analysis

Based on the electrical circuit of Fig. 7, the evaluation of the various power quantities within the engine can be performed. Starting from the left hand side of Fig. 7, the average expansion space power (AESP) for one cycle is:

$$P_{i-1}^e = - \oint p dV_e = \frac{1}{2} \text{Re}(p_{i-1}^e u_{i-1}^*) = \frac{1}{2} \text{Re}(p_i^e u_i^*) \quad (38)$$

In which u_{i-1}^* is the complex conjugate of the volumetric flow rate u_{i-1} and Re is the real part.

The same procedure can be used for the average compression space power (ACSP):

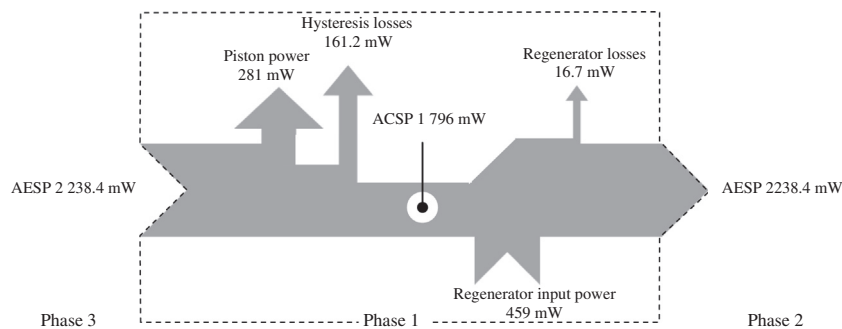


Fig. 9. Power flow diagram for the 3 phases Stirling engine.

$$P_i^c = - \oint p dV_c = \frac{1}{2} \text{Re}(p_i^c u_{i-1}^*) \quad (39)$$

For the predicted limit cycle ($U_p = 2.08$ mm at 57.35 Hz), the global power flow balance for the three-phase architecture (see Fig. 1a) is given in Fig. 9. The phases are numbered 1, 2 and 3.

The power flow diagram of Fig. 9 describes the input and output power within the engine. As the engines are connected to each other, the AESP of the phase 3 equals the AESP of the phase 1. The total heat input can be seen as the AESP plus the regenerator losses which have to be compensated. The regenerator input power is equal to the total power generated by the engine (piston power + hysteresis loss + regenerator losses) as can be seen in Fig. 9.

Therefore, the global efficiency is evaluated as:

$$\eta = \frac{281+161.2+16.7}{2238.4+16.7} = 20.35\%.$$

This value is the Carnot efficiency $\eta_{\text{Carnot}} = \frac{T_h - T_c}{T_h} = \frac{120 - 40}{293} = 20.35\%$ which is in accordance with the isothermal assumption used in the model.

The dissipated (useful) power at the piston of the engine 1 is 281 mW which is 12.46% of the input power.

The total losses sum up to 177.9 mW (7.89% of the input power) mostly associated to the gas spring hysteresis losses within the compression and expansion chambers (Eq. (31)). The ambient pressure operation, the low temperature conditions as well as the modest swept volume (31 cm³ for a total volume of 440 cm³) are in accordance with the small output power.

3.3. Thermodynamics theoretical validation

In order to perform a first validation of the proposed model, the Schmidt analysis is performed. This method is used in numerous studies [4,7] and its main results are recalled hereafter. From the ideal isothermal model assuming perfect gas law and given sinusoidal motions of the piston and the displacer, the power output can be expressed as in Eq. (40):

$$P_m = \omega p_{\text{mean}} V_{\text{sw}} \left(\frac{1}{2\pi} \frac{1}{2\beta} \frac{V_{\text{swe}}}{V_{\text{sw}}} \frac{\kappa(\tau - 1) \sin(\alpha)}{\sqrt{\tau^2 + \kappa^2 + 2\kappa\tau \cos(\alpha)}} (1 - \sqrt{1 - \beta^2}) \right) \quad (40)$$

In which τ , κ and β are defined in Eq. (41):

$$\begin{aligned} \tau &= \frac{T_c}{T_h} \\ \kappa &= \frac{V_{\text{swc}}}{V_{\text{swe}}} \\ \beta &= \frac{\sqrt{2\tau\kappa \cos(\alpha) + \tau^2 + \kappa^2}}{2V + \kappa + \tau} \end{aligned} \quad (41)$$

Table 2 presents the comparisons between the Schmidt analysis and the network model for the same parameters used in Subsection 3.2 (frequency and stroke U_p). No losses are considered in the Schmidt analysis. Therefore, the power evaluation used for comparison is the mean cycle power for one period. The results given in Table 2 clearly show that the two models matches.

3.4. Effect of the load

The engine operating conditions can be driven tuning the load applied to the engine. In the present model, a single damping

coefficient is used to model the dissipation and the load applied to the engine. We study here the impact of the two simultaneously.

For $T_h = 140$ °C, $T_c = 40$ °C, the effects load on the stroke and the frequency are studied for a damping coefficient varying between 0.4 N m⁻¹ s and 1.3 N m⁻¹ s. Fig. 10a and b present the evolution of the stroke and the frequency respectively.

From Fig. 10, it can be seen that low load cases lead to greater amplitudes and higher operating frequencies whereas the engine is slowed down with more moderate amplitude as the load is increased.

As expected, the output power is enhanced for low load which is clearly shown by Fig. 11a. However, in this case, the efficiency is reduced because of the increase of the losses. Fig. 11b plots the efficiency as a function of the damping. The efficiency is defined here as the power at the piston divided by the total heat input. Because the pressure drop losses lessen as the stroke decreased this efficiency increases with the damping.

These results have to be read carefully. Indeed, the temperatures of the heater and cooler are supposed to be constant for the whole operation range. Still, these temperatures are strongly dependant on performances of the heat exchangers.

Fig. 12 presents the evaluation of the Reynolds number for the fluid flow in the cooler, the heater and the regenerator. First, it can be noted that the results are in accordance with the proposed values of the Reynolds number in Fig. 3. Second, the variations of the Reynolds numbers for the heater and the cooler are significant and the associated heat exchange coefficients are likely to be modified accordingly. Consequently, for given external temperature conditions, the internal temperatures of the engine will change and so will do the operating variables.

The evolution of the temperatures as a function of the operating conditions (amplitude, frequency) and the geometrical parameters of the heat exchangers has been proposed in an analytical Stirling model in [21]. For a given predicted operation point, such a model could be used to refine the results. Therefore, an iterative modeling strategy could be performed. The proposed network model to account for the dynamic and the analytical Stirling model to take into account the heat exchanger performances can be used alternatively to converge to more accurate performance estimations.

4. Experimental analysis

A double acting three-phase Stirling engine has been built and is presented in Fig. 13. Its design is consistent with the details of Fig. 1. No pressurization is used and air is selected as the working gas. The heat input and cooling are not insulated to facilitate the instrumentation for the testing runs.

4.1. Experimental engine description

The heat source is obtained using 3×39 electric resistances connected in parallel. Therefore, 150 Watts heat power can be reached for each engine. However, in this first experimental testing, as no thermal isolation is used the heat power available for the engine is less than this value.

For one engine, two temperature sensors are fixed on the back-sides of the cold and hot parts ($T_{\text{COLD_exp}}$ and $T_{\text{HOT_exp}}$ respectively) whereas two other sensors are located within the compression and the expansion chambers ($T_{\text{c_exp}}$ and $T_{\text{e_exp}}$). Four additional temperature sensors are used to check the external temperatures of the two other engines. A pressure sensor is used and 3 accelerometers are fixed on the rigid links (see Fig. 13b). A constant electric power is used during the experiment whereas the liquid cooling flow rate is fixed.

Table 2
Comparisons between the linear network model and the equivalent Schmidt analysis.

	Schmidt analysis	Network model	Discrepancy (%)
Power (mW)	457.5	459	0.33
Heat input (mW)	2 247.4	2255.1	0.34
Frequency (Hz)	57.35 (given variable)	57.35	
Stroke (mm)	2.08 (given variable)	2.08	

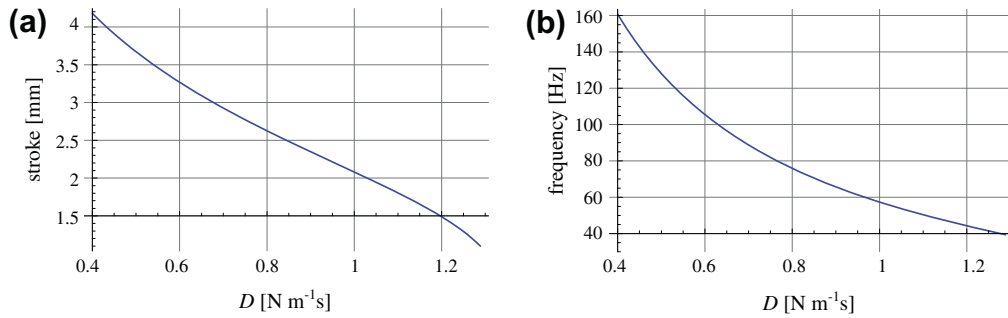


Fig. 10. (a) Evolution of the stroke with respect to the load, and (b) evolution of the frequency with respect to the load.

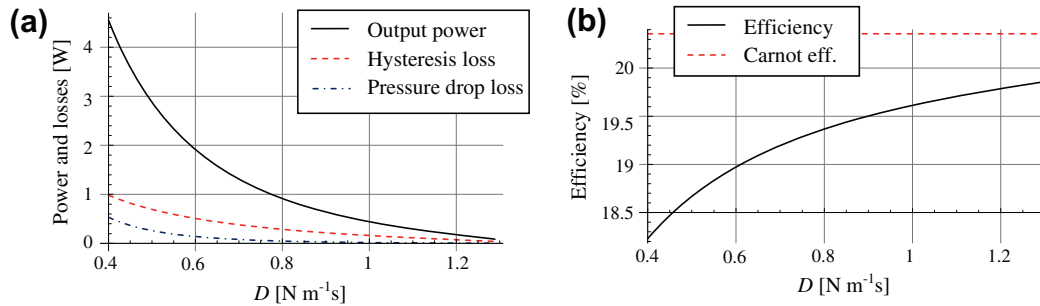


Fig. 11. (a) Output power and losses for various loads and (b) relative efficiency-power parametric curve.

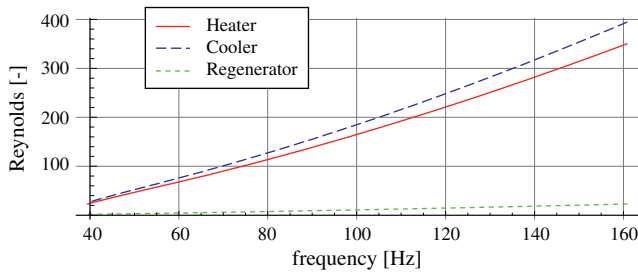


Fig. 12. Evaluation of the Reynolds numbers for the heat exchangers and the regenerator.

4.2. Experimental results and model comparison

Fig. 14a presents the experimental stroke of the phase 1 at $T_{\text{COLD_exp}} = 26.4^\circ\text{C}$, $T_{\text{c_exp}} = 22.1^\circ\text{C}$, $T_{\text{e_exp}} = 154.5^\circ\text{C}$ and $T_{\text{HOT_exp}} =$

161.1°C . The experimental operating frequency is 36.62 Hz whereas the stroke of the piston is evaluated at 0.15 mm.

The assumptions used for the model are listed hereafter:

- the temperatures of the model are set such as $T_e = T_{e_exp}$ and $T_c = T_{c_exp}$;
- we assume that the three phases geometry and external conditions are the same;
- the chambers of each engine are supposed to be symmetrical thus the volumes of the hot and cold chambers are equals;
- the pressure drop is fixed at four times its theoretical value to account for the misalignment of the regenerator wire mesh;
- the mass of the mobile component is estimated at 260 g;
- the damping effect is only related to the membrane and used as the tuning variable.

4.2.1. Steady state behavior

The damping coefficient D of the model is modified to reach the operating conditions (Eqs. (36) and (37)). The value is set to

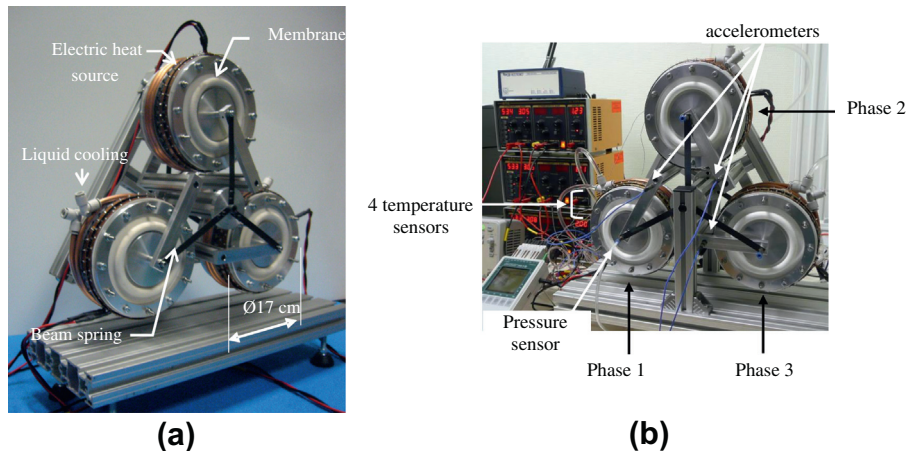


Fig. 13. Experimental prototype. (a) Global view and (b) instrumented engine.

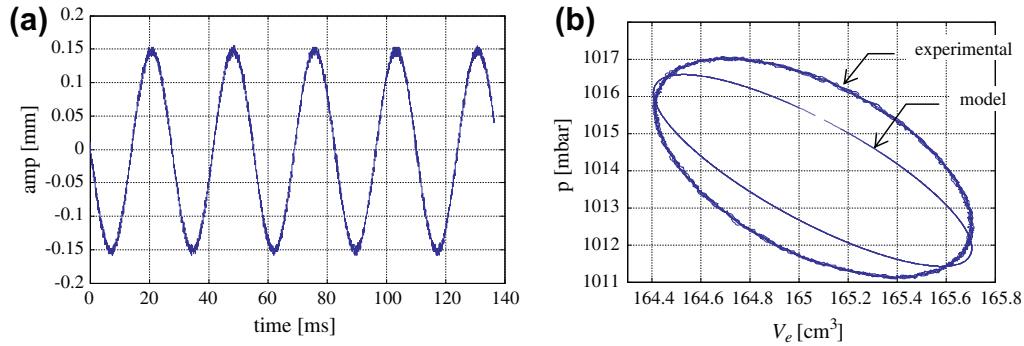


Fig. 14. (a) Amplitude of motion of the piston membrane of the engine 1 and (b) Clapeyron diagram for the expansion chamber of the engine 1 for $T_{c_exp} = 22.1$ °C and $T_{e_exp} = 154.5$ °C.

$4.33 \text{ Nm}^{-1} \text{ s}$ in which case the frequency is 36.42 Hz (0.56% difference with experimental result) and the stroke is 0.15 mm (1.74% difference with experimental result).

The experimental power at the expansion space (Fig. 14b) is 18.3 mW whereas for the model it is evaluated at 11.26 mW which is 38.6% difference. The kinematic behavior matches though the model estimation of the pressure amplitude is not so accurate.

These differences are thought to be associated to the symmetry assumption. Indeed, identified sealing flaws induced significant differences for the dead volumes of the three phases related to the different neutral positions of the membranes. Moreover, dissimilarities in the external temperatures have also been observed: phase 1 $T_{COLD_exp} = 26.4$ °C, $T_{HOT_exp} = 161.1$ °C; phase 2 $T_{COLD_exp} = 34.1$ °C, $T_{HOT_exp} = 165.0$ °C; phase 3 $T_{COLD_exp} = 31.2$ °C, $T_{HOT_exp} = 163.6$ °C.

In such a case, the expansion and compression swept volumes values are not the same for a given phase contrary to the model assumption. The greater measured pressure amplitude in Fig. 14b may be produced by a compression swept volume greater than the expansion swept volume.

4.2.2. Start-up conditions and self-starting property

The experimental start-up temperature is evaluated at $T_{c_exp} = 22.1$ °C, $T_{e_exp} = 145.1$ °C whereas, for the model at the previously determined damping coefficient is 154.4 °C. Modifying the value of the linear damping to $4.092 \text{ Nm}^{-1} \text{ s}$ allows the start-up temperature to be exactly estimated by the model. The difference between the damping coefficients is only 7.6%.

The experimental self-starting of the engine whenever the critical temperature is reached is presented in Fig. 15. The engine is first stopped by clamping the membrane of the engine 1. Then the membrane is released and no external excitation added. A

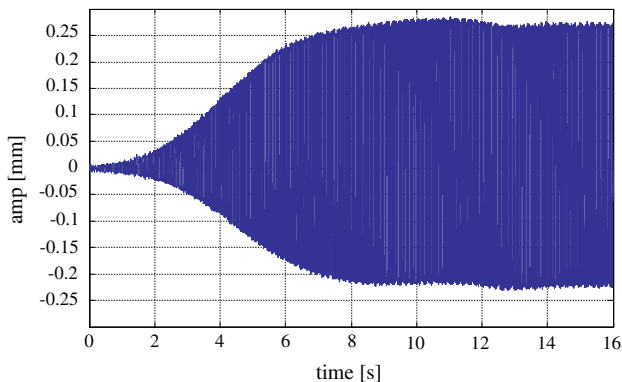


Fig. 15. Self-starting for $T_{c_exp} = 21.7$ °C and $T_{e_exp} = 156.7$ °C.

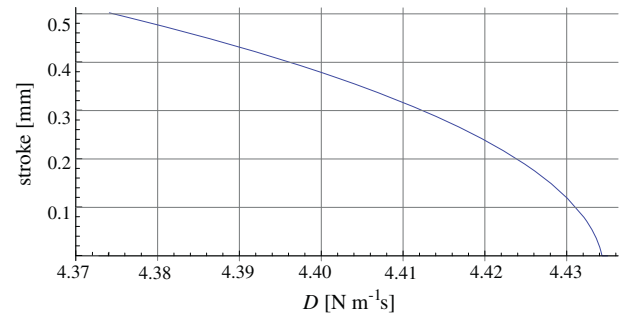


Fig. 16. Evolution of the stroke with respect to the damping coefficient.

motion occurs within a short delay and the amplitude rapidly increases to a steady state value. The final stroke is 0.26 mm for a frequency of 36.07 Hz.

It is worthy of note that in this case using the identified linear damping coefficient, the model gives a stroke of 0.66 mm for a frequency of 38.71 Hz. Increasing the damping coefficient allows to match the right amplitude. The required augmentation is only 2%.

In order to obtain a first assessment of the stroke sensitivity to the damping in the experimental conditions, Fig. 16 shows the evolution of the amplitude with respect to the damping coefficient for the experimental proposed parameters and $T_{c_exp} = 22.1$ °C, $T_{e_exp} = 154.5$ °C. It shows that for small strokes, the effect of the damping is the most important whereas it lessens for larger amplitude (i.e. smaller damping). This high sensitivity is consistent with the identification of the different damping coefficient and its drastic effect on the behavior of the engine.

4.2.3. Suggested strategies for optimization

The engine has then been modified to improve the sealing and reduce the dissymmetry between the phases: 6 new silicone membranes have been realized especially. Consequently, the moving mass has increased to 310 g so the thickness of the beam springs has been modified to keep an operating frequency of about 40 Hz. The new results are presented in Fig. 17. The experimental operating frequency is 39.98 Hz and the stroke adds up to 0.426 mm for a slightly higher temperature difference.

The new temperatures are set and the model is tuned adjusting the damping coefficient to $4.88 \text{ Nm}^{-1} \text{ s}$ (note that the dead volume have also been modified in the model to account for the new neutral positions of the membranes). Therefore, the theoretical frequency is 39.79 Hz (0.48% difference with experimental result) and the stroke is 0.424 mm (0.89% difference). The experimental power at the expansion space is 147.2 mW compared to the theoretical value of 108 mW. The power difference between the

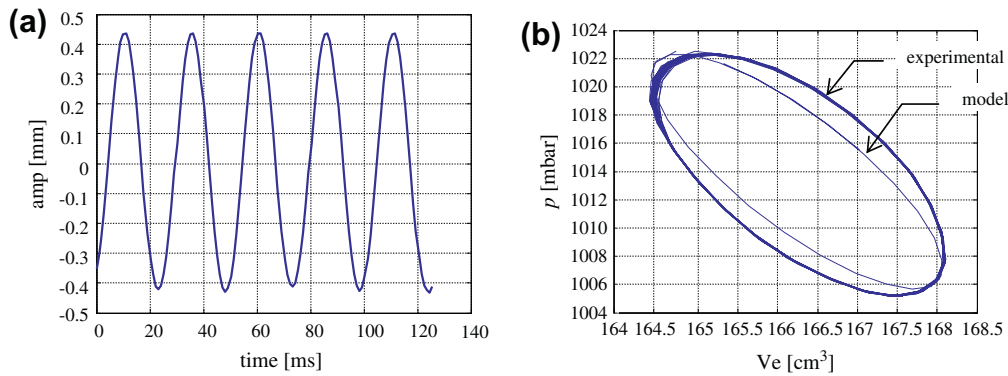


Fig. 17. (a) Amplitude of motion of the piston membrane of the engine 1 after modifications and (b) Clapeyron diagram for the expansion chamber of the engine 1 for $T_{c_exp} = 40.6\text{ }^{\circ}\text{C}$ and $T_{e_exp} = 178.2\text{ }^{\circ}\text{C}$.

experimental and theoretical results is reduced from 38.6% to 26.6%. It can also be noted that the output power has been drastically increased.

The improvement of the model prediction has been obtained while keeping the simplifying assumptions (symmetry of the engine, pressure drops in the heat exchangers neglected). With further modifications of the prototype to aim at a uniform heat distribution between the phases the model-experimental correlation is expected to be improved.

In the background of waste heat recovery, the performances of the engine would lie in efficiency and power output but also in standardization and economic criteria [3]. The proposed simple model will be used to define a preliminary design for a given application. Some of the considered evolutions to be achieved are presented hereafter and their effects will be easily assessed using the developed tools.

Further studies will aim at the characterization of the heat exchangers and their optimization. To account for the exchanger and the regenerator heat transfer performances, the network model (which integrates the friction losses) will be exploited in an iterative strategy in conjunction with a previously developed thermodynamics model [21]. The attractive solution which consists in the similarity of the hot and cold exchangers would also be assessed with respect to the performances gain with respect to standardization and cost reduction advantages.

In order to operate the engine at higher pressure, the mechanical connections will be modified. Indeed, the planar proposed architecture would favor the interface with the heat source but lead to a cantilevered arrangement of the mechanical connections. The present engine cannot stand pressures more than one bar without detrimental large deformations.

Additionally, the high nonlinear stiffness associated to the chosen beam structure would be lowered. The silicone membrane dissipation will also have to be reduced. By doing this, greater stroke amplitudes would be reached and larger swept volumes obtained eventually. The power of the engine will be enhanced accordingly providing that the pressure drop loss would be kept small.

5. Conclusion

The development of an equivalent electrical network model of a double acting FPSE has been detailed. The model allows the start-up condition to be established and the critical temperature evaluated. For given chambers temperatures and mean gas pressure, the steady state dynamic characteristics are estimated and the performances derived. A three-phase double acting free membrane engine architecture was proposed and is used as a test case. The amplitude of the engine is partly related to the non-linear

stiffening behavior of the beam spring suspension which has been modeled and introduced in the model. An experimental engine has been built. The tests confirm the self-starting capability of the engine. The experimental results were compared to the model which is representative enough for further design works. The optimization of the heat exchangers will aim at improving the performances. Additionally, the mechanical connections and silicone membranes will be modified to allow for higher pressure operation and minimize the mechanical dissipation and the stroke amplitude limitation. From the ensuing results, the economic viability of double acting FPSE for waste heat recovery applications will be appraised.

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